Appendix

Analytical solution for transient flow from a point source method

The conventional method for describing multidimensional infiltration and subsequent distribution of water in a bare soil is to use Richard's equation:

$$C(h)\frac{\partial h}{\partial t} = \nabla(K(h)\nabla H) \qquad \text{A. 1}$$

where $C(h)=d\theta/dh$ [L^{-1}] is the soil water capacity, $H=z+h$ [L] is the total hydraulic head, h [L] is the soil water pressure head, z is the vertical coordinate being positive upward, t [T] is time, K(h) [$L\ T^{-1}$] is the soil hydraulic conductivity and ∇ is the Laplacian (the spatial gradient) operator.

Analytical solution of the equation 1 for both steady state and transient water flow may be obtained by a linearization procedure using the exponential hydraulic conductivity function proposed by Gardner (1958):

$$K(h) = K_s e^{\alpha_{GRD}} \qquad \text{A. 2}$$

where K_s is the saturated hydraulic conductivity (LT^{-1}), $\alpha_{GRD} = 1/\lambda_{GRD}$ where λ_{GRD} is a scaling parameter which quantifies the importance of capillary forces relative to gravity. Also, analytical solutions requires calculation of the so-called matrix flux potential, ϕ, defined as (Philip, 1968):

$$\phi(h) = \int_{-\infty}^{h} K(h)dh = \frac{K(h)}{\alpha_{GRD}} \qquad \text{A. 3}$$

Warrick (1974) solved the Richards equation analytically by using similar transformations (Eqs. A. 1 and A. 3) coupled with the additional assumption that $dK/d\theta = k$ or $d\theta/d\phi = \alpha_{GRD}/k$, where k is a constant, to linearize Richards Equation:

$$\frac{\partial \phi}{\partial t} = \frac{k}{\alpha_{GRD}} \nabla^2 \phi - k \frac{\partial \phi}{\partial z} \qquad \text{A. 4}$$

To solve Eq. A. 4 analytically, the dimensionless variables: $R = \alpha_{\mathrm{GRD}} r/2$, $Z = \alpha_{\mathrm{GRD}} z/2$, $T = \alpha_{GRD}\ kt/4$, $\rho = \sqrt{R^2 + Z^2}$, and the dimensionless matric flux potential: $\Phi_B = \alpha q \phi / 8\pi$ were introduced, where r and z are spatial radial and vertical coordinates, and t is time. With the initial condition $\phi(r, z, 0) = 0$ and the boundary conditions $-\frac{\partial \phi}{\partial z} + \alpha_{GRD} \phi = 0$ for $\mathrm{z} = 0, r \neq 0$, the analytical solution for a buried point source in an infinite medium is given as (Warrick, 1974):

$$\Phi_B(\mathrm{R,Z,T}) = \frac{e^z}{2\rho}\left[e^{\rho}\ erfc\ \left(\frac{\rho}{2\sqrt{T}} + \sqrt{T}\right) + e^{\rho}\ erfc\ \left(\frac{\rho}{2\sqrt{T}} - \sqrt{T}\right)\right] \qquad \text{A. 5}$$

where $erfc$ is the complementary error function given as (Spiegel and Liu, 1999):

$$erfc\ (x) = 1 - erfc\ (x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2}\ du \qquad \text{A. 6}$$

The solution for a surface point source is:

$$\Phi_S(R, Z, T) = 2\left[\Phi_B - e^{2Z} \int_z^{\infty} e^{-2Z'} (\Phi_B)_{Z=Z'} dZ\right] \qquad \text{A. 7}$$

the integration of Eq. A. 7 can be accomplished by using the Gauss-Leguerre quadrature (Sen et al.,

1992) :

$$\int_0^\infty e^{-2Z'} (\Phi_B)_{Z=Z'}\, dZ' = e^{-2Z} \int_0^\infty e^{-x} (\Phi_B)_{Z\prime=Z+x/2} \frac{dx}{2} \qquad \text{A. 8}$$

$$= \frac{1}{2} e^{-2Z} \sum_{i=0}^{x} \omega_i \, (\Phi_B)_{Z\prime=Z+x/2}$$

where Z' = Z+x/2. The weights ω_i and the sampling points x_i (for the 15-point formula used in this study) may be obtained from Carnahan *et al.* (1969).

For regular cyclic inputs (i.e., irrigation cycles) or other temporal variations in source strength, the value of Φ is obtained by superposition in time and knowing that $\Phi_B = \alpha q \phi / 8\pi$ (Warrick, 1974):

$$\phi(R, Z, T) = \frac{\alpha}{8\pi} \sum_{i=0}^{n} (q_i - q_{i-1})\ \Phi(R, Z, T - T_i) \qquad q_{-1} = 0, T_0 = 0, T > T_i \qquad \text{A. 9}$$

Pressure head values can then be obtained from Eqs. A. 2 and A. 3 as:

$$h(r, z, t) = \frac{1}{\alpha_{GRD}} ln \left(\frac{\alpha_{GRD} \phi(r, z, t)}{K_s} \right) \qquad \text{A. 10}$$

Corresponding transient soil water content values $\theta(r, z, t)$ may be obtained by the soil water retention model proposed by Russo (1988):

$$S_e = \frac{\theta - \theta_r}{\theta_s - \theta_r} = [\exp{(-0.5\alpha_{GR} h)} (1 + 0.5\alpha_{GR} h)]^{\left(\frac{2}{\mu_R + 2}\right)} \qquad \text{A. 11}$$

where α_{GR} is the soil parameter appearing in the Gardner's model for hydraulic conductivity related to the pore size distribution, while μ_R is a parameter related to tortuosity. *Se* is effective saturation and θ_s and θ_r are the water contents at h=0 and for $h \rightarrow \infty$, respectively. The choice of the Russo model comes from the fact that it is appropriate for the linearized equations as it is based on the same parameter α_{GR} used in the Gardner's exponential hydraulic conductivity function.